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Constructive Description of the Compact Set of States

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We show a construction of the set of all pure states of any physical system. This set appears to be compact in a physically meaningful topology. The approach is based on quantum logic notions, but is "constructive" in the sense that we assume that our knowledge about a system increases when we perform more and more experiments and that the set of states is not given to us from the very beginning but is determined by our knowledge at any stage of investigations. The set of all states is obtained when all possible experiments are performed, which may require an infinite number of experiments. In such a situation the set of all states exists only on an abstract, purely theoretical level but nevertheless by our construction it is still compact.

1. INTRODUCTION

In a previous paper (Posiewnik, 1985a) one of us presented a construction of a set of physical states based on the Scott (Scott, 1982) information systems theory. Sets of pure states obtained by this construction appeared to be compact in some physically natural topology, which in turn implied that the set of all (pure and mixed) states was a compact convex subset of a locally convex Hausdorff real topological vector space (Posiewnik, 1985b). Such an embedding of the set of all physical states was the ultimate goal and the cause of all efforts since once we have it we may make the full use of the Choquet theory (Phelps, 1966; Alfsen, 1971; Asimow and Ellis, 1980). Particularly, we may describe decompositions of a mixed state into its pure components using integrals with respect to measures concentrated on the set of pure states (Pykacz, 1983; Posiewnik and Pykacz, 1981). In the present paper we propose another construction of the set of all pure states of a physical system which yields the same results. The present approach is more specific since it is based on the notion of lattice of properties which is a

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particular example of an information system (Posiewnik, 1985a) but is "constructive," i.e., we do not assume that the whole set of states is given from the very beginning but we construct it step by step by making more and more experiments on the physical system. In this point our research strategy agrees in some sense with that of Finkelstein (1969, 1972), von Weizsäcker (1975, 1977, 1979), Bastin (1976), etc.

We adopt to our description the quantum logic approach in the form advocated mostly by Piron (1976) and Aerts (1981, 1982). Since a detailed exposition of this approach can be found in the papers mentioned above, we remind the reader here only of some basic notions which are necessary in our construction. According to Piron and Aerts one gains information about a physical system by asking "questions," i.e., by performing experiments whose outcomes can be interpreted in terms of "yes" or "no." A question is said to be "true" iff, when one would decide to ask it, the answer "yes" would come out with certainty. We say that the question α is stronger than the question β and denote it by $\alpha < \beta$ if whenever α is true then also β is true. This relation is a preorder relation since, although it is reflexive and transitive, generally $\alpha < \beta$ and $\beta < \alpha$ does not imply $\alpha = \beta$. Nevertheless such questions are "equivalent" in the sense that by asking them we study features of physical systems which always appear together. The relation $\approx$ defined on the set of questions by a formula

$$\alpha \approx \beta \text{ iff } \alpha < \beta \text{ and } \beta < \alpha$$

is an equivalence relation and its classes of equivalence are called "properties." If a question α belongs to the equivalence class a, we say that α tests the property a. The preorder relation on the set of questions defines in a natural way a partial order relation on the set of properties: Let a and b be properties; then

$$a < b \text{ iff } \alpha < \beta \text{ for all } \alpha \in a \text{ and } \beta \in b$$

We denote by I the "trivial property," i.e., the equivalence class of questions which are always true and by 0 the equivalence class of questions which are never true. The set $\mathscr{L}$ of all properties is a lattice with I as the top and 0 as the bottom element.

The meet of the family of properties $\{a_i\}$ is a property $\bigwedge_i a_i$ obtained in the following way: we take one question α_i from every equivalence class a_i and we construct a "product question" $\prod_i \alpha_i$ by choosing as we want, at random or not, one of α_i and according to $\prod_i \alpha_i$ the answer obtained for the chosen question. The property $\bigwedge_i a_i$ is a property tested by $\prod_i \alpha_i$, i.e., it is the equivalence class of $\prod_i \alpha_i$. The join of the family of properties $\{a_i\}$ is a property $\bigvee_i a_i$ defined by the following formula:

$$\bigvee_i a_i = \bigwedge_{a_j < b} b, \qquad b \in \mathscr{L}$$

The property a of a physical system is called "actual" iff any question which tests a, i.e., which belongs to a, is true. A state of a physical system is defined as a set of all actual properties or, if we want to study a physical system using only questions, as a set of all questions which are true. A state of a system defined in such a way is a pure state in the sense of Dirac (Dirac, 1958) since by specifying it we obtain maximal information about the system. A state of a system can change in time because when time elapses properties which were not actual may become actual and vice versa. By the set of all (pure) states of a physical system we mean the set of all states which can be achieved by a system under all kinds of external conditions possible to apply.

2. THE CONSTRUCTION

A reader of the majority of papers on axiomatical foundations of physical theories usually gets an impression that the whole knowledge about all mathematical structures which describe a physical system is either given to him from the very beginning or that it is at least possible to obtain it in full details. We do not want to discuss here the philosophical question if these mathematical structures exist on their own—although we are eager to agree that they do—but we want to check carefully the way in which we gain knowledge about them. The Piron–Aerts quantum logic approach, which deals with "yes-no" experiments and in which the notion of a property and a state emerges from this more primitive notion of question, is particularly suited for this task. Since we do not assume that when we start to study a physical system we have already any knowledge about it, it should be possible to repeat experiments many times using sequences of equally prepared copies of a system to establish the preorder relation between questions. This possibility of obtaining sufficiently many copies of a system is necessary also for the reason that we cannot exclude the situation that some experiments could destroy the system.

Let us now check carefully what we must do to build up a lattice of properties and recognize which sets of properties represent states of a given physical system. We ask questions, i.e., we perform "yes-no" experiments; then, guided by results of experiments, we introduce preorder relation on questions, construct product questions, and group questions into equivalence classes to obtain properties (Finkelstein, 1979). Next we introduce partial order relation on properties and finally we check which sets of properties consist of properties which for some sequences of equally prepared copies of the system are actual together so they represent states of the system. But the natural way to study a physical system is not to ask all possible questions at once (which is usually even impossible) but rather to ask questions one by one and build up partial models at every stage of investigations. In such a way we obtain a sequence of models which reflects a process of gaining information about a system. Every lattice of properties contains all previously built up lattices as sublattices and we can find remainders of every state of a system in all previously obtained lattices at least in the form of the trivial property I which is always actual. On the other hand each set of properties which was recognized as a state of a system after asking some questions can give rise to a not necessarily single state in the more detailed model built up after asking more questions. Let us illustrate these ideas in a simple case of a system consisting of a large number of molecules H_2O. Each stage of investigations will be indexed by the number of "generating questions," i.e. questions which, together with an "inverse" operation (forming from a question α a question α' by interchanging outcomes "yes" and "no") and the "product" operation, are sufficient to obtain the whole lattice of properties. For the sake of simplicity we will not list those product questions which will be equivalent to questions which are already listed.

$n = 1$

Questions: l_1 = "Does the system exist?", l_2 = "Do we have anything to study?", l_1', l_2'.
Relations on questions: $l_1' \approx l_2' < l_1 \approx l_2$.
Properties: $I \ni l_1, l_2$; $0 \ni l_1', l_2'$.
Relations on properties: $0 < I$.

$n = 2$

New questions: α_1 = "Does the system has structure of crystal?", α_2 = "Does the system has definite shape?", α_1', α_2'.
Relations on questions: $l_1' \approx l_2' < \alpha_1' \approx \alpha_2'$, $\alpha_1 \approx \alpha_2 < l_1 \approx l_2$.
Properties: I; $a \ni \alpha_1, \alpha_2$; $a' \ni \alpha_1', \alpha_2'$; 0.
Relations on properties: $0 < a$, $a' < I$.

$n = 3$

New questions: β = "Does the system has definite volume?", β', $\alpha_i' \cdot \beta$, $(\alpha_i \cdot \beta')'$; $i = 1, 2$.
Relations on questions: $l_i' < \alpha_i < \beta$, $(\alpha_i \cdot \beta')' < l_i$; $l_i' < \alpha_i' \cdot \beta < \alpha_i'$, $\beta < l_i$; $l_i' < \beta' < (\alpha_i \cdot \beta')'$, $\alpha_i' < l_i$; $i = 1, 2$
Properties: I, a, a', $b \ni \beta$, $b' \ni \beta'$, $a' \wedge b \ni \alpha_i' \cdot \beta$, $a \vee b' \ni (\alpha_i \cdot \beta')'$, 0.
Relations on properties: $0 < a < b$, $a \vee b' < I$; $0 < a' \wedge b < a'$, $b < I$; $0 < b' < a'$, $a \vee b' < I$.

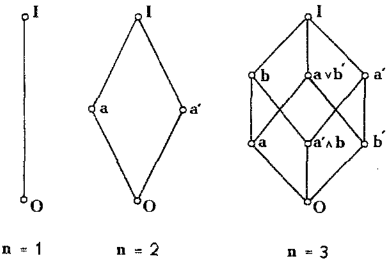

Fig. 1.

The Hasse diagrams of lattices of properties for $n = 1, 2, 3$ are drawn on the Figure 1.

Let us assume that our samples of water are investigated under normal pressure but some of them were prepared to have temperature −10°C, some +10°C and some +110°C. Then, for $n = 3$, we observe that the whole set of samples splits itself into three subsets for which the following properties are actual: for $t = -10$°C properties a, b, $a \vee b'$, I; for $t = +10$°C properties a', b, $a' \wedge b$, I, and for $t = +110$°C properties b', a', $a \vee b'$, I. We can easily recognize these three sets of properties as representing solid, liquid, and gaseous states of water, respectively. If we stop investigations on the stage $n = 2$ then we can recognize only a solid and a fluid state represented by sets $\{a, I\}$ and $\{a', I\}$, respectively. A fluid state gives rise to a liquid and to a gas when we add new question β, while a solid state remains unsplit.

3. TOPOLOGY

Since we shall treat a set of all states of a physical system as a limit of an inverse system of topological spaces we remind the necessary notions. A reader is referred to the book of Engelking (1977) for a more detailed exposition of this useful notion. An inverse system of topological spaces is a family $S = \{X_\sigma, f_\rho^\sigma, \Sigma\}$, where Σ is a set directed by a relation $\leq$, X_σ is a topological space for any $\sigma \in \Sigma$, and for any $\rho \leq \sigma$ f_ρ^σ is a continuous mapping of X_σ into X_ρ such that the following conditions are satisfied:

$$f_\tau^\rho f_\rho^\sigma = f_\tau^\sigma \qquad \text{for any } \tau \leq \rho \leq \sigma$$

$$f_\sigma^\sigma = id_{X_\sigma} \qquad \text{for any } \sigma \in \Sigma$$

An element $\{x_\sigma\}$ of the Cartesian product $\prod_{\sigma \in \Sigma} X_\sigma$ is called a thread of S if $f_\rho^\sigma(x_\sigma) = x_\rho$ for any $\rho \leq \sigma$. A limit of the inverse system S is the subspace of $\prod_{\sigma\in\Sigma} X_\sigma$ consisting of all threads of S. It is usually denoted by $\varprojlim S$. If in the directed set Σ there exists an element σ_0 such that $\sigma \leq \sigma_0$ for every $\sigma \in \Sigma$ then the limit of S is homeomorphic to the space X_{σ_0}. Compactness of topological spaces is hereditary under forming limits of inverse systems, precisely the limit of an inverse system of nonempty compact spaces is compact and nonempty.

The very construction of sequence of more and more detailed models which was presented in the previous section provides a possibility for using a limit of an inverse system for describing the set of all states despite of the fact that we cannot usually get the whole information about it in any finite sequence of experiments. The directed set Σ which is needed consists of natural numbers which were used for indexing models together with the ordinary partial order. If S_m and S_n are sets of states obtained on the mth and nth stage of investigations and if a state $p \in S_n$ consists of properties $a_1, \ldots, a_k$, then we define the image of p by the mapping f^n_m as a set $f^n_m(p) = p \cap \mathcal{L}_m$, i.e., $f^n_m(p)$ is a state of a physical system which consists of these properties from the set p which were already recognized on the mth stage of investigations. Obviously $f^n_n = id_{S_n}$ for any n and $f^m_l f^n_m = f^n_l$ for any $l \leq m \leq n$ so the only thing which remains to fulfill the requirement of a definition of an inverse system is continuity of all mappings f^n_m. But let us notice that any set of states S_n which is obtained after performing finite sequence of experiments is by the very construction finite so if we equip S_n with any Hausdorff (or even T_1) topology it becomes a discrete topological space for which all mappings f^n_m are continuous. The assumption that any physically meaningful topology on the set of states should be Hausdorff is not a strong one and it can be often found in the literature (Cole, 1968; Gunson, 1967).

Let us assume for a moment that a given physical system is of such a nature that all its properties can be recognized in a finite sequence of experiments. In such a situation in the set of numbers Σ used for indexing models there exists the greatest number n_0 and by the already mentioned theorem the limit $\varprojlim S$ of an inverse system of topological spaces $S = \{S_n, f^n_m, \Sigma\}$ is homeomorphic to the space S_{n_0} whose elements represent all possible states of a physical system. For any two numbers $m, n \in \Sigma$ iff $m \leq n \leq n_0$ then the partial model S_n approximates S_{n_0} better than S_m. Of course the situation when Σ is finite is exceptional but we see from it that even for infinite Σ elements of a limit of the inverse system of topological spaces $S = \{S_n, f^n_m, \Sigma\}$ represent states of a physical system which could have been achieved experimentally if we were able to perform infinite sequence of experiments. Therefore we can call any element $p \in S_n$ an "experimental" or "epistemological" state while any element of $\varprojlim S$ should be called an "abstract" or "theoretical" or "ontological" state of a physical system. We are thus led to a form of platonic dualism consisting of an ideal world of theoretical states in contrast to an empirical world of real states. Since all sets S_n of experimental states are finite they are compact in any Hausdorff topology (which is as well in this case discrete). The set of all theoretical states $\varprojlim S$ is therefore compact although its topology, when Σ is not finite, need not be discrete any more. We find here Brouwer's idea of the *esprit de finesse* (which confines itself to finite objects "constructed" by limited means) as alien to the *esprit de géométrie* (which describes theoretical or "potential" situation and the "true" nature of the system).

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